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Quantum Knizhnik–Zamolodchikov equation for $U_q(\widehat{\mathfrak{sl}}_n)$ and integral formula

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Abstract. An integral formula for solutions to the level-zero quantum Knizhnik–Zamolodchikov equation associated with the vector representation of $U_q(\widehat{\mathfrak{sl}}_n)$ is presented. This formula gives a generalization of both our previous work for $U_q(\widehat{\mathfrak{sl}}_2)$ and Smirnov's formula for form factors of the $SU(n)$ chiral Gross–Neveu model.

1. Introduction

In our previous paper [1], we gave an integral formula for solutions to the quantum Knizhnik–Zamolodchikov (KZ) equation [2] for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ when the spin is $\frac{1}{2}$, the level is zero and $|q| < 1$. The present paper is a $U_q(\widehat{\mathfrak{sl}}_n)$ generalization of [1]. Our approach is based on [3]. Instead of solving the quantum KZ equation, we consider a system of difference equations for a vector-valued function in N variables $(z_1, \ldots, z_N)$ which takes values in the N-fold tensor product of the vector representation $V = \mathbb{C}^n$ of $U_q(\widehat{\mathfrak{sl}}_n)$.

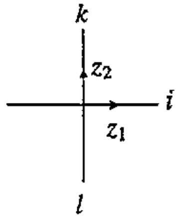
For a fixed complex number q satisfying $0 < |q| < 1$, let $R(z) \in \mathrm{End}(V \otimes V)$ be the standard trigonometric R-matrix associated with the vector representation $V \cong \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n$ of $U_q(\widehat{\mathfrak{sl}}_n)$. The matrix $R(z)$ satisfying the Yang–Baxter equation and the unitarity relation is

$$R(z) v_{\varepsilon_1'} \otimes v_{\varepsilon_2'} = \sum_{\varepsilon_1, \varepsilon_2} v_{\varepsilon_1} \otimes v_{\varepsilon_2} R^{\varepsilon_1 \varepsilon_2}_{\varepsilon_1' \varepsilon_2'}(z)$$

where the non-zero entries are

$$\begin{aligned}
R^{\varepsilon\varepsilon}_{\varepsilon\varepsilon}(z) &= 1 \\
R^{\varepsilon\varepsilon'}_{\varepsilon\varepsilon'}(z) &= b(z) = \frac{(1-z)q}{1-zq^2} \qquad \text{if } \varepsilon \neq \varepsilon' \\
R^{\varepsilon\varepsilon'}_{\varepsilon'\varepsilon}(z) &= \begin{cases} \dfrac{(1-q^2)z}{1-zq^2} & \text{if } \varepsilon < \varepsilon' \\[2ex] \dfrac{(1-q^2)}{1-zq^2} & \text{if } \varepsilon > \varepsilon'. \end{cases}
\end{aligned}$$

In statistical-mechanics language, each entry of the R-matrix is a local Boltzmann weight for a single vertex with bond states $i, j, k, l \in \mathbb{Z}_n$:

$R^{ik}_{jl}(z_1/z_2) =$ 

where each line carries a spectral parameter.

In what follows, we shall work with the tensor product of finitely many V's. Following the usual convention, we let $R_{jk}(z)$ ($j \neq k$) signify the operator on $V^{\otimes N}$ acting as $R(z)$ on the (j, k)th tensor components and as an identity on the other components. In particular, we have $R_{kj}(z) = P_{jk}R_{jk}(z)P_{jk}$ where $P \in \mathrm{End}(V \otimes V)$ stands for the transposition $P(x \otimes y) = y \otimes x$.

The equations we are concerned with in this paper are those for (1) R-matrix symmetry and (2) deformed cyclicity for a function $G(z_1, \ldots, z_N) \in V^{\otimes N}$:

$$(1) \qquad P_{j\,j+1}G(\ldots, z_{j+1}, z_j, \ldots) = R_{j\,j+1}(z_j/z_{j+1})G(\ldots, z_j, z_{j+1}, \ldots) \tag{1.1}$$

$$(2) \qquad P_{12}\cdots P_{N-1N}G(z_2, \cdots, z_N, z_1q^{-2n}) = D_1G(z_1, \ldots, z_N). \tag{1.2}$$

In (1.2), D_1 is an operator acting on the first component as $D = \mathrm{diag}(\delta_1, \ldots, \delta_n)$, the entries of which will be specified below, and an identity acting on the other components. These are two of the axioms that form factors in integrable models should satisfy [4]. Smirnov [4] also pointed out that (1.1) and (1.2) imply the level-zero quantum KZ equation [2]

$$\begin{aligned} G(z_1, \ldots, z_jq^{2n}, \ldots, z_N) &= R_{j-1\,j}(z_{j-1}/z_jq^{2n})^{-1}\ldots R_{1\,j}(z_1/z_jq^{2n})^{-1}D_j^{-1} \\ &\quad \times R_{j\,N}(z_j/z_N)\ldots R_{j\,j+1}(z_j/z_{j+1})G(z_1, \ldots, z_j, \ldots, z_N). \end{aligned} \tag{1.3}$$

Throughout this paper, the functions we consider are not necessarily single valued in z_j but are meromorphic in the variable $\log z_j$. Accordingly, the shift $z_j \to z_jq^{-2n}$ as in (1.2) is understood to mean $\log z_j \to \log z_j - 2n\log q$.

In the following, we set $\tau = q^{-1}$. Define the components of G by

$$G(z_1, \ldots, z_N) = \sum_{\varepsilon_j=1}^{n} v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_N} G^{\varepsilon_1\cdots\varepsilon_N}(z_1, \ldots, z_N). \tag{1.4}$$

Then equation (1.1) reads as

$$G^{\cdots\varepsilon\varepsilon\cdots}(\cdots, z_j, z_{j+1}, \cdots) = G^{\cdots\varepsilon\varepsilon\cdots}(\cdots, z_{j+1}, z_j, \cdots) \tag{1.5}$$

$$\begin{aligned} G^{\cdots\varepsilon\varepsilon'\cdots}(\ldots, z_j, z_{j+1}, \ldots) &= \frac{z_j - z_{j+1}\tau^2}{(z_j - z_{j+1})\tau}G^{\cdots\varepsilon'\varepsilon\cdots}(\ldots, z_{j+1}, z_j, \ldots) \\ &\quad - \frac{(1-\tau^2)z^{\varepsilon\varepsilon'}_{jj+1}}{(z_j - z_{j+1})\tau}G^{\cdots\varepsilon'\varepsilon\cdots}(\ldots, z_j, z_{j+1}, \ldots) \end{aligned} \tag{1.6}$$

where

$$z^{\varepsilon\varepsilon'}_{jj+1} = \begin{cases} z_j & \text{if } \varepsilon < \varepsilon' \\ z_{j+1} & \text{if } \varepsilon > \varepsilon' \end{cases}$$

and (1.2) reads as

$$G^{\varepsilon_2\cdots\varepsilon_N\varepsilon_1}(z_2, \ldots, z_N, z_1\tau^{2n}) = \delta_{\varepsilon_1}G^{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}(z_1, z_2, \ldots, z_N). \tag{1.7}$$

Note that the singularity at $z_j = z_{j+1}$ in (1.6) is spurious. Equations (1.5)–(1.7) split into blocks, each involving components such that

$$\sharp\{j|\varepsilon_j = i\} = m_i \qquad N = \sum_{i=1}^{n} m_i.$$

In the present paper, we restrict ourselves to the case $m_1 = \cdots = m_n = m$ and, hence, $N = mn$. According to this restriction, we set $\delta_i = \tau^{m(1-n)+2(1-i)}$.

We use the abbreviation $z^{(j)} = (z_1^{(j)}, \ldots, z_m^{(j)})$. Consider the extreme component

$$G^{\overbrace{n\cdots n}^{m}\cdots\overbrace{1\cdots 1}^{m}}(z^{(n)}|\cdots|z_1^{(1)}) = H(z^{(n)}|\cdots|z^{(1)}). \tag{1.8}$$

Because of (1.5), this function is symmetric in the variables $z^{(1)}, \ldots, z^{(n)}$, separately. Equation (1.6) tells us that all the components with fixed m are uniquely determined from H. Conversely, given any such H, the Yang–Baxter equation guarantees that (1.1) can be solved consistently under condition (1.8).

We wish to find an integral formula of the form

$$H(z_1, \ldots, z_N) = (S_{MN}F)(z_1, \ldots, z_N) \tag{1.9}$$

where S_{MN} stands for the following integral transform:

$$\begin{aligned}(S_{MN}F)&(z_1, \ldots, z_N)\\ &= \oint_C \mathrm{d}x_1 \cdots \oint_C \mathrm{d}x_M\, F(x_1, \ldots, x_M|z_1, \ldots, z_N)\Psi(x_1, \ldots, x_M|z_1, \ldots, z_N).\end{aligned} \tag{1.10}$$

The notation is explained below.

The kernel Ψ has the form

$$\Psi(x_1, \ldots, x_M|z_1, \ldots, z_N) = \vartheta(x_1, \ldots, x_M|z_1, \ldots, z_N) \prod_{\mu=1}^{M}\prod_{j=1}^{N} \psi\left(\frac{x_\mu}{z_j}\right)$$

where

$$\psi(z) = \frac{1}{(zq^{n-1}; q^{2n})_\infty (z^{-1}q^{n-1}; q^{2n})_\infty} \qquad (z; p)_\infty = \prod_{k=0}^{\infty}(1 - zp^k).$$

Assume that the function ϑ is antisymmetric and holomorphic in the $x_\mu \in \mathbb{C}\backslash\{0\}$, is symmetric and meromorphic in $\log z_j \in \mathbb{C}$ and possesses the following transformation property:

$$\begin{aligned}\vartheta(x_1, \ldots, x_M|z_1, \ldots, z_j\tau^{2n}, \ldots, z_N) &= \vartheta(x_1, \ldots, x_M|z_1, \ldots, z_N) \prod_{\mu=1}^{M} \frac{-z_j\tau^{n-1}}{x_\mu}\\ \vartheta(x_1, \ldots, x_\mu\tau^{2n}, \ldots, x_M|z_1, \cdots, z_N) &= \vartheta(x_1, \ldots, x_M|z_1, \ldots, z_N) \prod_{j=1}^{N} \frac{-x_\mu\tau^{n-1}}{z_j}.\end{aligned} \tag{1.11}$$

The function ϑ is otherwise arbitrary and the choice of ϑ's corresponds to that of solutions. The integration $\oint_C \mathrm{d}x_\mu$ is along a simple closed curve $C = C(z_1, \ldots, z_N)$

oriented anticlockwise, which encircles the points $z_j\tau^{-n+1-2nk}$ $(1 \leqslant j \leqslant N, k \geqslant 0)$ but not $z_j\tau^{n-1+2nk}$ $(1 \leqslant j \leqslant N, k \geqslant 0)$. Finally,

$$F(x_1, \ldots, x_M|z^{(n)}|\ldots|z^{(1)}) = \frac{\Delta^{(m)}(x_1, \ldots, x_M|z^{(n)}|\cdots|z^{(1)})}{\prod_{\substack{k,k'=1\\k<k'}}^{n}\prod_{j=1}^{m}\prod_{j'=1}^{m}(z_j^{(k)} - z_{j'}^{(k')}\tau^2)} \tag{1.12}$$

where $\Delta^{(m)}$ is a certain homogeneous polynomial yet to be determined which is antisymmetric in the variables $(x_1, \ldots, x_n)$ and symmetric in the variables $z^{(1)}, \ldots, z^{(n)}$, separately.

The rest of the paper is organized as follows. In section 2, we introduce a special basis of V in terms of the quantum monodromy operators. In section 3, we describe the main theorem of the present paper. The subsequent two sections are devoted to the proof of our main theorem; section 4 for $m = 1$ and section 5 for the general case. In section 6, we discuss the relation between other works and our own.

2. Quantum monodromy operators and the special basis

We shall construct a special basis $\{w_{\alpha_1\cdots\alpha_N}(z_1, \ldots, z_N)\}$ with $\alpha_j \in \{1, 2, \ldots, n\}$ of $V^{\otimes N}$ depending on the parameter $(z_1, \ldots, z_N)$, which satisfies

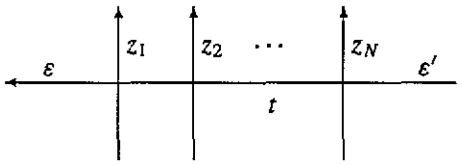
$$P_{jj+1}w_{\cdots\alpha_{j+1}\alpha_j\cdots}(\ldots, z_{j+1}, z_j, \ldots) = R_{jj+1}(z_j/z_{j+1})w_{\cdots\alpha_j\alpha_{j+1}\cdots}(\ldots, z_j, z_{j+1}, \ldots). \tag{2.1}$$

The procedure is as follows. Define the quantum monodromy operator $\mathcal{T}_{\varepsilon\varepsilon'}(z_1, \ldots, z_N|t) \in \mathrm{End}(V^{\otimes N})$ by

$$R_{1\,N+1}(z_1/t)\cdots R_{N\,N+1}(z_N/t) = (\mathcal{T}_{\varepsilon\varepsilon'}(z_1, \cdots, z_N|t))_{1\leqslant\varepsilon,\varepsilon'\leqslant n}. \tag{2.2}$$

Here the $n \times n$ matrix structure is defined relative to the base $v_1, \ldots, v_n$ of the $(N + 1)$th tensor component of $V^{\otimes(N+1)}$

$$\mathcal{T}_{\varepsilon\varepsilon'}(z_1, \cdots, z_N|t) =$$

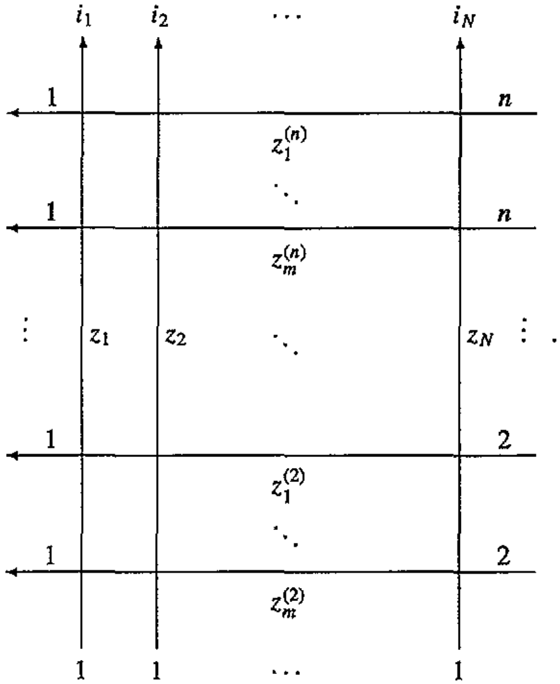


For $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_j \in \{1, \ldots, n\}$, set $J_i^\alpha = \{j|\alpha_j = i\}$ and $\{z_1^{(i)}, \ldots, z_m^{(i)}\} = \{z_j|\alpha_j = i\}$, where $1 \leqslant i \leqslant n$, and set

$$\begin{aligned} w_\alpha(z_1, \cdots, z_N) &= \prod_{l=1}^{m}\mathcal{T}_{1n}(z_1, \ldots, z_N|z_l^{(n)})\cdots\prod_{l=1}^{m}\mathcal{T}_{12}(z_1, \ldots, z_N|z_l^{(2)})\Omega \\ \Omega &= v_1 \otimes \cdots \otimes v_1 \qquad \in V^{\otimes N}. \end{aligned} \tag{2.3}$$

Then $w_\alpha(z_1, \ldots, z_N)$ is visualized as follows.

$$w_\alpha(z_1,\cdots,z_N) = \sum v_{i_1}\otimes\cdots\otimes v_{i_N}$$



Let us introduce the ordered indices. Set $(\alpha_1,\ldots,\alpha_N) > (\varepsilon_1,\ldots,\varepsilon_N)$ if and only if $\alpha_i=\varepsilon_i$, $(1\leqslant i\leqslant k)$ and $\alpha_{k+1}>\varepsilon_{k+1}$. Define the components of the basis by

$$w_{\alpha_1\cdots\alpha_N}(z_1,\ldots,z_N)=\sum_{\varepsilon_1,\ldots,\varepsilon_N} v_{\varepsilon_1}\otimes\cdots\otimes v_{\varepsilon_N} w^{\varepsilon_1\cdots\varepsilon_N}_{\alpha_1\cdots\alpha_N}(z_1,\ldots,z_N) \tag{2.4}$$

then we have

$$w^{\varepsilon_1\cdots\varepsilon_N}_{\alpha_1\cdots\alpha_N}(z_1,\ldots,z_N)=0 \qquad \text{if } (\alpha_1,\ldots,\alpha_N)<(\varepsilon_1,\ldots,\varepsilon_N). \tag{2.5}$$

Furthermore, for $\beta=(n,\ldots,n,\ldots,2,\ldots,2,1,\ldots,1)$, we have

$$w^\beta_\beta(z_1,\ldots,z_N)=\prod_{\substack{k\in J_i^\beta, l\in J_j^\beta\\ i<j}} b(z_k/z_l). \tag{2.6}$$

From (2.1), (2.5) and (2.6), we obtain the following explicit formula:

$$G(z_1,\ldots,z_N)=\sum_\alpha w_\alpha(z_1,\ldots,z_N)H(\{z_j\}_{j\in J_n^\alpha}|\cdots|\{z_j\}_{j\in J_1^\alpha})\prod_{\alpha_i<\alpha_j}\frac{1}{b(z_i/z_j)}.$$

For $2\leqslant p\leqslant n$ and $n\geqslant i_1>\cdots>i_p\geqslant 1$, set

$$v^{(i_1\cdots i_p)}=\sum_{\sigma\in\mathfrak{S}_p}(-\tau)^{l(\sigma)}v_{\sigma(i_1)}\otimes\cdots\otimes v_{\sigma(i_p)}$$

where $l(\sigma)$ is the minimum number of permutations such that

$$v_{\sigma(i_1)}\otimes\cdots\otimes v_{\sigma(i_p)}=\left(\prod P_{jj+1}\right)v_{i_1}\otimes\cdots\otimes v_{i_p}.$$

For example

$$
\begin{aligned}
v^{(21)} &= v_2 \otimes v_1 - \tau v_1 \otimes v_2 \\
v^{(321)} &= v_3 \otimes v_2 \otimes v_1 - \tau(v_3 \otimes v_1 \otimes v_2 + v_2 \otimes v_3 \otimes v_1) \\
&\quad + \tau^2(v_1 \otimes v_3 \otimes v_2 + v_2 \otimes v_1 \otimes v_3) - \tau^3 v_1 \otimes v_2 \otimes v_3 .
\end{aligned}
$$

For $i \in \{i_1, \ldots, i_p\}$, one can easily check the following formulae:

$$
R_{1p+1}(z) R_{2p+1}(z\tau^2) \cdots R_{pp+1}(z\tau^{2p-2}) v^{(i_1 \cdots i_p)} \otimes v_i = \prod_{j=1}^{p-1} b(z\tau^{2j}) v^{(i_1 \cdots i_p)} \otimes v_i . \tag{2.7}
$$

The poles of $w_\alpha(z)$ exist only at $z_j = z_i \tau^2$ for $i < j$ and $\alpha_i > \alpha_j$. Thus, we have

$$
\begin{aligned}
&\mathrm{Res}_{z_2 = z_1 \tau^2} \cdots \mathrm{Res}_{z_n = z_{n-1}\tau^2}\, w_{\alpha_1 \cdots \alpha_n}(z_1, \ldots, z_n) \\
&\qquad = \delta_{\alpha_1 n} \delta_{\alpha_2 n-1} \cdots \delta_{\alpha_n 1} z_1^{n-1} (\tau^2 - 1)^{n-1} \tau^{(n-1)^2} [n-1]! \;\; v^{(n \cdots 21)}
\end{aligned} \tag{2.8}
$$

where

$$
[k]! = [k] \cdots [1] \qquad [k] = \frac{\tau^k - \tau^{-k}}{\tau - \tau^{-1}} .
$$

Furthermore, we obtain the recursive residue formula

$$
\begin{aligned}
\mathrm{Res}_{z_2 = z_1 \tau^2} \cdots \mathrm{Res}_{z_n = z_{n-1}\tau^2}\, w_{\alpha_1 \cdots \alpha_N}(z_1, \ldots, z_N) &= \prod_{\substack{\alpha_i > \alpha_j \\ i \leqslant n < j}} b(z_j / z_i) \\
&\quad \times (\mathrm{Res}_{z_2 = z_1 \tau^2} \cdots \mathrm{Res}_{z_n = z_{n-1}\tau^2}\, w_{n \cdots 1}(z_1, \ldots, z_n)) \otimes w_{\alpha_{n+1} \cdots \alpha_N}(z_{n+1}, \ldots, z_N)
\end{aligned} \tag{2.9}
$$

for $\alpha_i = n + 1 - i$ $(1 \leqslant i \leqslant n)$. By combining (2.1) and (2.9) and using (2.7), we have the useful expression

$$
\begin{aligned}
&\mathrm{Res}_{z_m^{(n-1)} = z_m^{(n)} \tau^2} \cdots \mathrm{Res}_{z_m^{(1)} = z_m^{(2)} \tau^2}\, w_\alpha^{\overbrace{n \cdots n}^{m} \cdots \hat{i} \cdots \hat{i} \cdots \overbrace{1 \cdots 1}^{m} \overbrace{i \cdots i}^{m}} (z^{(n)} | \cdots | z^{(1)}) \\
&\qquad = \prod_{k=2}^{n} \prod_{l=1}^{m-1} \prod_{j=n+1-k}^{n-1} b(z_m^{(n)} \tau^{2j} / z_l^{(k)}) \prod_{\substack{\alpha_{mk+l} < \alpha_{jm} \\ 1 \leqslant l < m}} b(z_l^{(n-k)} / z_m^{(j)}) \\
&\qquad\quad \times (z_m^{(n)})^{n-1} (-\tau)^{i-1} (\tau^2 - 1)^{n-1} \tau^{(n-1)^2} [n-1]! \\
&\qquad\quad \times v^{(n \cdots 21)} \otimes w_{\alpha'}^{\overbrace{n \cdots n}^{m-1} \cdots \hat{i} \cdots \hat{i} \cdots \overbrace{1 \cdots 1}^{m-1} \overbrace{i \cdots i}^{m-1}} (z'^{(n)} | \cdots | z'^{(1)})
\end{aligned} \tag{2.10}
$$

for $\alpha' = (\alpha_1, \ldots, \widehat{\alpha_m}, \ldots, \widehat{\alpha_{2m}}, \ldots, \alpha_{nm-1})$ and $\alpha_{mi} = n + 1 - i$ $(1 \leqslant i \leqslant n)$. Here we use the abbreviation $z'^{(i)} = (z_1^{(i)}, \ldots, z_{m-1}^{(i)})$.

3. Main theorem

Now we present the main theorem of the present paper. In what follows, we use the abbreviations

$$
\begin{aligned}
z^{(j)} &= (z_1^{(j)}, \ldots, z_{m-1}^{(j)}, z_m^{(j)}) = (z'^{(j)}, z_m^{(j)}) \\
z^{(j)} \tau^{\pm 1} &= (z_1^{(j)} \tau^{\pm 1}, \ldots, z_{m-1}^{(j)} \tau^{\pm 1}, z_m^{(j)} \tau^{\pm 1}) = (z'^{(j)} \tau^{\pm 1}, z_m^{(j)} \tau^{\pm 1}) \\
x &= (x_1, \ldots, x_M) .
\end{aligned}
$$

The polynomial $\Delta^{(m)}$ in (1.12) is given by

$$\Delta^{(m)}(x|z^{(n)}|\cdots|z^{(2)}|z^{(1)}) = \det(A_\lambda^{(m)}(x_\mu|z^{(n)}|\cdots|z^{(2)}|z^{(1)}))_{1\leqslant\lambda,\mu\leqslant M} \tag{3.1}$$

where $M = M_m = (n-1)m - 1$ and $N = mn$. The entries of the $M \times M$ matrix $A^{(m)}$ are defined as follows. Let us introduce the polynomial

$$f_\lambda^{(N)}(y|z_1,\dots,z_N) = \sum_{\kappa=0}^{\lambda-1}(-1)^\kappa((y\tau)^{\lambda-\kappa} - (y\tau^{-1})^{\lambda-\kappa})\sigma_\kappa(z_1,\dots,z_N)$$

where $\sigma_\kappa(z_1,\dots,z_n)$ denotes the κth elementary symmetric polynomials

$$\prod_{j=1}^{n}(t+z_j) = \sum_{\kappa=0}^{n}\sigma_\kappa(z_1,\dots,z_n)t^{n-\kappa}.$$

Note that for $\alpha > 0$

$$f_{N+\alpha}^{(N)}(y|z_1,\dots,z_N) = y^\alpha\left\{\tau^\alpha\prod_{j=1}^{N}(y\tau - z_j) - \tau^{-\alpha}\prod_{j=1}^{N}(y\tau^{-1} - z_j)\right\}. \tag{3.2}$$

Define the polynomial

$$\begin{aligned} A_\lambda^{(m)}(x|z^{(n)}|\cdots|z^{(1)}) &= \sum_{k=1}^{n}\prod_{j=1}^{m}(x - z_j^{(k)}\tau^{2k-n-1}) \\ &\quad\times\frac{1}{x}f_\lambda^{((n-1)m)}\left(x\tau^{n+1-2k}\,\middle|\,z^{(n)}\tau,\dots,z^{(k+1)}\tau\overset{k}{\wedge}z^{(k-1)}\tau^{-1},\dots,z^{(1)}\tau^{-1}\right). \end{aligned} \tag{3.3}$$

This is a homogeneous polynomial of degree $m+\lambda-1$, symmetric with respect to the $z^{(k)}$'s for each k, separately. By constructions (3.1) and (3.3), $\Delta^{(m)}$ is a homogeneous polynomial of degree $M_m m + M_m(M_m+1)/2$ with the correct symmetries. For $n = 3$, it reads as

$$\begin{aligned} A_\lambda^{(m)}(x|z^{(3)}|z^{(2)}|z^{(1)}) &= \prod_{j=1}^{m}(x - z_j^{(3)}\tau^2)\frac{1}{x}f_\lambda^{(2m)}(x\tau^{-2}|z^{(2)}\tau^{-1}, z^{(1)}\tau^{-1}) \\ &\quad+\prod_{j=1}^{m}(x - z_j^{(2)})\frac{1}{x}f_\lambda^{(2m)}(x|z^{(3)}\tau, z^{(1)}\tau^{-1}) \\ &\quad+\prod_{j=1}^{m}(x - z_j^{(1)}\tau^{-2})\frac{1}{x}f_\lambda^{(2m)}(x\tau^2|z^{(3)}\tau, z^{(2)}\tau). \end{aligned}$$

The following is the main theorem of this paper.

Theorem 3.1. The integral formula

$$G(z_1,\dots,z_N) = \sum_\alpha w_\alpha(z_1,\dots,z_N)H(\{z_j\}_{j\in J_n^\alpha}|\cdots|\{z_j\}_{j\in J_1^\alpha})\prod_{\alpha_i<\alpha_j}\frac{1}{b(z_i/z_j)}$$

satisfies (1.1) and (1.2) with $(\delta_1,\delta_2,\dots,\delta_n) = (\tau^{-(n-1)m}, \tau^{-(n-1)m-2},\dots,\tau^{-(n-1)(m+2)})$ where H is defined by (1.9)–(1.12) and $w_\alpha(z_1,\dots,z_N)$ is defined by (2.2)–(2.3), respectively.

First of all, we note the following lemmas.

Lemma 3.2. The following holds

$$(S_{MN}\Delta^{(m)})(z^{(n)}|\cdots|z^{(2)}|z'^{(1)}, z_0\tau^{2n}) = (S_{MN}\tilde{\Delta}^{(m)})(x|z^{(n)}|\cdots|z^{(2)}|z'^{(1)}, z_0)$$

where

$$\tilde{\Delta}^{(m)}(x|z^{(n)}|\ldots|z^{(2)}|z'^{(1)}, z_0) = \det(\tilde{A_\lambda}^{(m)}(x_\mu|z^{(n)}|\cdots|z^{(2)}|z'^{(1)}, z_0))_{1\leqslant\lambda,\mu\leqslant M}$$

and

$$\begin{aligned}\tilde{A}_\lambda^{(m)}(x|z^{(n)}|\cdots|z^{(2)}|z'^{(1)}, z_0) &= \frac{x - z_0\tau^{n-1}}{x - z_0\tau^{n+1}} A_\lambda^{(m)}(x|z^{(n)}|\cdots|z^{(2)}|z'^{(1)}, z_0\tau^{2n}) \\ &\quad + \frac{z_0\tau^{n+1}}{x}\left\{\tau^{-2}\prod_{j=1}^{N-1}\frac{x\tau^{n-1} - z_j}{z_0\tau^2 - z_j} - \frac{x - z_0\tau^{n-1}}{x - z_0\tau^{n+1}}\prod_{j=1}^{N-1}\frac{x\tau^{-n+1} - z_j}{z_0\tau^2 - z_j}\right\} \\ &\quad \times A_\lambda^{(m)}(z_0\tau^{n+1}|z^{(n)}|\cdots|z^{(2)}|z'^{(1)}, z_0\tau^{2n}).\end{aligned}$$

This can be proved in a similar manner to that used for $n = 2$, see [1].

To prove theorem 3.1, it is sufficient to show (1.7) for n cases; i.e.

$$\begin{aligned}&G^{\overbrace{n\cdots n}^{m}\cdots\hat{i}\cdots\hat{i}\cdots\overbrace{1\cdots 1}^{m}\overbrace{i\cdots i}^{m}}(z^{(n)}|\cdots|z'^{(1)}, z_0\tau^{2n}) \\ &\qquad = \delta_i G^{i\,\overbrace{n\cdots n}^{m}\cdots\hat{i}\cdots\hat{i}\cdots\overbrace{1\cdots 1}^{m}\overbrace{i\cdots i}^{m-1}}(z_0|z^{(n)}|\cdots|z'^{(1)}) \qquad (3.4)\end{aligned}$$

for $i = 1, \ldots, n$.

Let $z_j^{(k)} = z_{(n-k)m+j}$ for $n \geqslant k \geqslant 2$ and $z_j'^{(1)} = z_{(n-1)m+j}$. Let us define $J_i(j_1, \ldots, j_{i-1})$ recursively as follows

$$\begin{aligned} J_1 &= \{0, 1, \ldots, m\} \ni j_1 \\ J_2(j_1) &= \{j_1, m+1, \ldots, 2m\} \ni j_2 \\ &\vdots \\ J_{n-2}(j_1, \ldots, j_{n-3}) &= \{j_{n-3}, (n-3)m+1, \ldots, (n-2)m\} \ni j_{n-2} \\ J_{n-1}(j_1, \ldots, j_{n-2}) &= \{j_{n-2}, (n-2)m+1, \ldots, (n-1)m\} \ni j_{n-1}. \end{aligned} \qquad (3.5)$$

Set

$$\begin{aligned}\varphi^{(m)}(j_1, \ldots, j_{n-1}) &= \prod_{k_1\in J_1\backslash\{j_1\}}\frac{z_{k_1} - z_0\tau^2}{(z_{j_1} - z_{k_1})\tau}\prod_{k_2\in J_2\backslash\{j_2\}}\frac{z_{k_2} - z_{j_1}\tau^2}{(z_{j_2} - z_{k_2})\tau}\cdots \\ &\quad \times \prod_{k_{n-1}\in J_{n-1}\backslash\{j_{n-1}\}}\frac{z_{k_{n-1}} - z_{j_{n-2}}\tau^2}{(z_{j_{n-1}} - z_{k_{n-1}})\tau}\prod_{j=1}^{m-1}\frac{z_j'^{(1)} - z_{j_{n-1}}\tau^2}{(z_j'^{(1)} - z_0)\tau^2}.\end{aligned}$$

Equation (3.4), for $i = 1$ and $\delta_1 = \tau^{m(1-n)}$, is satisfied if the following proposition holds.

Proposition 3.3.

$$\begin{aligned}&\sum_J \varphi^{(m)}(j_1, \ldots, j_{n-1})\Delta^{(m)}(x|\overset{j_1}{\hat{z_{j_1}}}|\cdots|\overset{j_{n-1}}{\hat{z_{j_{n-1}}}}|z_{j_{n-1}}, z'^{(1)}) \\ &\qquad = (-\tau)^{m(1-n)}\prod_{k=2}^{n}\prod_{j=1}^{m}\frac{z_j^{(k)} - z_0\tau^2}{z_j^{(k)} - z_0\tau^{2n-2}}\tilde{\Delta}(x|z^{(n)}|\cdots|z^{(2)}|z'^{(1)}, z_0). \qquad (3.6)\end{aligned}$$

In section 4, we prove theorem 3.1 for $m = 1$. In section 5, we verify proposition 3.3 and also show that (3.4), for $i \neq 1$, reduces to proposition 3.3, which implies theorem 3.1 for any m.

4. The $m = 1$ case

In this section, we prove theorem 3.1 for $m = 1$. Notice that for $m = 1$, $A_\lambda^{(m)}(x|z)$ coincides with

$$A_\lambda^{(1)}(x|z_1, \ldots, z_n) = \sum_{\kappa=0}^{\lambda}(-1)^\kappa x^{\lambda-\kappa}(\tau^{n(\lambda-\kappa)+\kappa} - \tau^{-n(\lambda-\kappa)-\kappa})\sigma_\kappa(z_1, \ldots, z_n) \tag{4.1}$$

and that it is linear with respect to the z's. In this case, $z_i = z_1^{(n+1-i)}$. In this section, we use the abbreviations $\Delta^{(1)} = \Delta$, $\tilde{\Delta}^{(1)} = \tilde{\Delta}$ and $A_\lambda^{(1)} = A_\lambda$. Since the polynomial $A_\lambda(x|z_1, \ldots, z_n)$ is symmetric with respect to the variable $(z_1, \ldots, z_n)$, by using (1.6), we obtain

$$\begin{aligned} G(z_1, \ldots, z_n)^{n\cdots\hat{i}\cdots 1i} &= (-\tau)^{1-i}H(z_1, \ldots, z_n) \\ G(z_1, \ldots, z_n)^{in\cdots\hat{i}\cdots 1} &= (-\tau)^{i-1}H(z_1, \ldots, z_n). \end{aligned} \tag{4.2}$$

Thus

$$G(z_2, \ldots, z_n, z_1\tau^{2n})^{n\cdots 21} = \delta_1 G(z_1, \ldots, z_n)^{1n\cdots 2} \qquad \delta_1 = \tau^{1-n} \tag{4.3}$$

implies

$$G(z_2, \ldots, z_n, z_1\tau^{2n})^{n\cdots\hat{i}\cdots 1i} = \delta_i G(z_1, \ldots, z_n)^{in\cdots\hat{i}\cdots 1} \qquad \delta_i = \delta_1\tau^{2-2i}. \tag{4.4}$$

Consequently, to prove theorem 3.1 for $m = 1$, it is enough to show (4.3).

Now we prepare the following lemmas.

Lemma 4.1.

$$\tilde{\Delta}(x_1, \ldots, x_{n-2}|z_1, \ldots, z_n)|_{z_j = z_1\tau^{2n-2}} = 0 \qquad (j = 2, 3, \ldots, n). \tag{4.5}$$

Proof. Note that

$$\sum_{\lambda=0}^{n-3} z_1^\lambda A_{n-2-\lambda}(x|z_1\tau^{-1}, z_1\tau, z_3, \ldots, z_n)$$
$$= \frac{1}{x}\left\{(x\tau^{n-2} - z_1)\prod_{j=3}^{n}(x - z_j\tau^{-n+1}) - (x\tau^{-n+2} - z_1)\prod_{j=3}^{n}(x - z_j\tau^{n-1})\right\}.$$

Hence, we obtain the linear dependence

$$\sum_{\lambda=0}^{n-3}(z_1\tau^{2n-1})^\lambda \tilde{A}_{n-2-\lambda}(x|z_1, z_1\tau^{2n-2}, z_3, \ldots, z_n) = 0$$

which implies (4.5). □

Lemma 4.2.

$$\prod_{\mu=2}^{n-2}(x_\mu - z_1\tau)\Delta(z_1\tau^{-1}, x_2, \ldots, x_{n-2}|z_1\tau^{-n}, z_2, \ldots, z_n)$$
$$= \prod_{\mu=2}^{n-2}(x_\mu - z_1\tau^{-1})\Delta(z_1\tau, x_2, \ldots, x_{n-2}|z_1\tau^n, z_2, \ldots, z_n). \tag{4.6}$$

Proof. Let us first show that

$$\det(C_{\lambda,\mu})_{1\leqslant\mu,\lambda\leqslant n-2} = \det(C'_{\lambda,\mu})_{1\leqslant\mu,\lambda\leqslant n-2} \tag{4.7}$$

where

$$C_{\lambda\mu} = \begin{cases} A_\lambda(z_1\tau^{-1}|z_1\tau^{-n}, z_2, \ldots, z_n) & \text{if } \mu = 1 \\ (x_\mu - z_1\tau)A_\lambda(x_\mu|z_1\tau^{-n}, z_2, \ldots, z_n) & \text{if } \mu \neq 1 \end{cases}$$

$$C'_{\lambda\mu} = \begin{cases} A_\lambda(z_1\tau|z_1\tau^{n}, z_2, \ldots, z_n) & \text{if } \mu = 1 \\ (x_\mu - z_1\tau^{-1})A_\lambda(x_\mu|z_1\tau^{n}, z_2, \ldots, z_n) & \text{if } \mu \neq 1. \end{cases}$$

Perform the following elementary transformations to the matrix $(C_{\lambda\mu})_{1\leqslant\lambda,\mu\leqslant n-2}$

(1) (ith row) $- z_1\tau^{n-1}$(($i-1$)st row) $\quad (i = 2, \ldots, n-2)$

(2) (ith column) $+ z_1\tau$(($i-1$)st column) $\quad (i = 2, \ldots, n-2)$

and to the matrix $(C'_{\lambda\mu})_{1\leqslant\lambda,\mu\leqslant n-2}$

(1) (ith row) $- z_1\tau^{-n+1}$(($i-1$)st row) $\quad (i = 2, \ldots, n-2)$

(2) (ith column) $+ z_1\tau^{-1}$(($i-1$)st column) $\quad (i = 2, \ldots, n-2)$.

Then we have (4.7) and, therefore, (4.6). □

Because the polynomial $A_\lambda(z_1, \ldots, z_n)$ is symmetric with respect to $(z_1, \ldots, z_n)$, proposition 3.3 holds if the following proposition holds:

Proposition 4.3.

$$\tilde{\Delta}(x_1, \ldots, x_{n-2}|z_1, \ldots, z_n) = \prod_{j=2}^{n} \frac{z_j - z_1\tau^{2n-2}}{z_j - z_1\tau^2} \Delta(x_1, \ldots, x_{n-2}|z_1, \ldots, z_n). \tag{4.8}$$

Proof. Due to the linearity of the determinant, we have

$$\begin{aligned} \tilde{\Delta}(x_1, \ldots, x_{n-2}|z_1, \ldots, z_n) &= \prod_{\mu=1}^{n-2} \frac{x_\mu - z_1\tau^{n-1}}{x_\mu - z_1\tau^{n+1}} \Delta(x_1, \ldots, x_{n-2}|z_1, \ldots, z_n) \\ &\quad + \sum_{\nu=1}^{n-2} \prod_{\substack{\mu=1 \\ \mu\neq\nu}}^{n-2} \frac{x_\mu - z_1\tau^{n-1}}{x_\mu - z_1\tau^{n+1}} g(x_\nu|z_1|z_2, \ldots, z_n) \\ &\quad \times \Delta(x_1, \ldots, \overset{\nu}{\widehat{z_1\tau^{n+1}}} \cdots x_{n-2}|z_1\tau^{2n}, z_2, \ldots, z_n) \end{aligned} \tag{4.9}$$

where

$$g(x|z_1|z_2, \ldots, z_n) = \frac{z_1\tau^{n+1}}{x} \left\{ \tau^{-2} \prod_{j=2}^{n} \frac{x\tau^{n-1} - z_j}{z_1\tau^2 - z_j} - \frac{x - z_1\tau^{n-1}}{x - z_1\tau^{n+1}} \prod_{j=2}^{n} \frac{x\tau^{-n+1} - z_j}{z_1\tau^2 - z_j} \right\}.$$

Hence, we get

$$\begin{aligned} &\tilde{\Delta}(z_1\tau^{n-1}, x_2, \ldots, x_{n-2}|z_1, \ldots, z_n) \\ &\quad = \prod_{j=2}^{n} \frac{z_j - z_1\tau^{2n-2}}{z_j - z_1\tau^2} \prod_{\mu=2}^{n-2} \frac{x_\mu - z_1\tau^{n-1}}{x_\mu - z_1\tau^{n+1}} \Delta(z_1\tau^{n+1}, x_2, \ldots, x_{n-2}|z_1\tau^{2n}, \ldots, z_n). \end{aligned} \tag{4.10}$$

From lemma 4.2 and (4.10), we obtain

$$\tilde{\Delta}(z_1\tau^{n-1}, x_2, \ldots, x_{n-2}|z_1, \ldots, z_n) = \prod_{j=2}^{n} \frac{z_j - z_1\tau^{2n-2}}{z_j - z_1\tau^2} \Delta(z_1\tau^{n-1}, x_2, \ldots, x_{n-2}|z_1, \ldots, z_n). \tag{4.11}$$

The polynomials $\tilde{\Delta}$ and Δ have the same factor $\prod_{1\leqslant\mu\leqslant\nu\leqslant n-2}(x_\mu - x_\nu)$. Furthermore, the degrees of $\tilde{\Delta}$ and Δ with respect to x_μ are at most $(n-2)$. Thus, we get

$$\begin{aligned}\tilde{\Delta}(x_1, x_2, \ldots, x_{n-2}|z_1, \ldots, z_n) - \prod_{j=2}^{n} \frac{z_j - z_1\tau^{2n-2}}{z_j - z_1\tau^2} \Delta(z_1\tau^{n-1}, x_2, \ldots, x_{n-2}|z_1, \ldots, z_n) \\ = c(z_1, \ldots, z_n) \prod_{1\leqslant\mu<\nu\leqslant n-2} (x_\mu - x_\nu) \prod_{\mu=1}^{n-2} (x_\mu - z_1\tau^{n-1})\end{aligned} \tag{4.12}$$

where $c(z_1, \ldots, z_n)$ is a homogeneous rational function of $(z_1, \ldots, z_n)$ with total degree 0. From lemma 4.1, c has zeros at $z_j = z_1\tau^{2n-2}$ and may have poles at only $z_j = z_1\tau^2$. Thus, c must have the form

$$c(z_1, \ldots, z_n) = s \prod_{j=2}^{n} \frac{z_j - z_1\tau^{2n-2}}{z_j - z_1\tau^2} \qquad s \in \mathbb{C}.$$

By comparing both sides of (4.12) at $z_1 = 0$, we obtain $s = 0$. □

Therefore, by taking symmetry (4.2) into account, theorem 3.1 for $m = 1$ is proved.

5. General case

In this section, we prove theorem 3.1 for the general case. Let $I_l = \{(\gamma_1, \ldots, \gamma_l)| n \geqslant \gamma_1 > \cdots \gamma_l \geqslant 1\}$ for $1 \leqslant l \leqslant n-1$. Fix non-negative integers $m(\gamma_1, \ldots, \gamma_l)$ for $1 \leqslant l \leqslant n-1$ and set

$$N_k = \sum_{l=1}^{n-1} \sum_{\substack{(\gamma_1,\ldots,\gamma_l)\in I_l \\ \gamma_l \neq k}} m(\gamma_1, \ldots, \gamma_l).$$

In what follows, we often use the abbreviation $\gamma = (\gamma_1, \ldots, \gamma_l)$. Define the polynomial with respect to variables x and $\zeta^{(\gamma)} = (\zeta_1^{(\gamma)}, \ldots, \zeta_{m(\gamma)}^{(\gamma)})$ by

$$\begin{aligned}&A_\lambda(x|\zeta^{(n)}|\cdots|\zeta^{(1)}|\zeta^{(nn-1)}|\cdots|\zeta^{(21)}|\cdots|\zeta^{(\gamma_1\cdots\gamma_l)}|\cdots|\zeta^{(n-1\cdots 21)}) \\ &\quad = \sum_{k=1}^{n} \prod_{l=1}^{n-1} \prod_{\substack{(\gamma)\in I_l, \\ \gamma_l = k}} \prod_{j=1}^{m(\gamma)} (x - \zeta_j^{(\gamma)}\tau^{2k-n-3+2i}) \\ &\qquad \times \frac{1}{x} f_\lambda^{(N_k)}\left(x\tau^{n+1-2k} \middle| \bigcup_{l=1}^{n-1} \bigcup_{\substack{(\gamma)\in I_l, \\ \gamma_l > k > \gamma_{l+1}}} \bigcup_{j=1}^{m(\gamma)} \zeta_j^{(\gamma)}\tau^{2i-1}\right).\end{aligned} \tag{5.1}$$

For $n = 3$, this reads as

$$
\begin{aligned}
xA_\lambda(x|\zeta^{(3)}|\zeta^{(2)}|\zeta^{(1)}|\zeta^{(32)}|\zeta^{(31)}|\zeta^{(21)}) = {} & \prod_{j=1}^{m(3)}(x-\zeta_j^{(3)}\tau^2)\prod_{j=1}^{m(3,2)}(x-\zeta_j^{(32)}\tau^2) \\
& \times \prod_{j=1}^{m(3,1)}(x-\zeta_j^{(31)}\tau^2) f_\lambda^{(N_3)}(x\tau^{-2}|\zeta^{(2)}\tau^{-1},\zeta^{(1)}\tau^{-1},\zeta^{(21)}\tau^{-1}) \\
& + \prod_{j=1}^{m(2)}(x-\zeta_j^{(2)})\prod_{j=1}^{m(3,2)}(x-\zeta_j^{(32)}\tau^2) \\
& \times \prod_{j=1}^{m(2,1)}(x-\zeta_j^{(21)}) f_\lambda^{(N_2)}(x|\zeta^{(3)}\tau,\zeta^{(1)}\tau^{-1},\zeta^{(31)}\tau) \\
& + \prod_{j=1}^{m(1)}(x-\zeta_j^{(1)}\tau^{-2})\prod_{j=1}^{m(3,1)}(x-\zeta_j^{(31)}) \\
& \times \prod_{j=1}^{m(2,1)}(x-\zeta_j^{(21)}) f_\lambda^{(N_1)}(x\tau^2|\zeta^{(3)}\tau,\zeta^{(2)}\tau,\zeta^{(32)}\tau^3).
\end{aligned}
$$

From the recursion relation

$$
f_\lambda^{(N+1)}(y|z_1,\ldots,z_N,a) = f_\lambda^{(N)}(y|z_1,\ldots,z_N) - af_{\lambda-1}^{(N)}(y|z_1,\ldots,z_N)
$$

it follows that the polynomial $A_\lambda^{(m)}$ satisfies

$$
\begin{aligned}
& A_\lambda^{(m)}(x|z^{(n)}|\cdots|z^{(1)})|_{\bigcup_{l=1}^{n-1}\bigcup_{j=1}^{l} z^{(\gamma_j)}=\zeta^{(\gamma_1\cdots\gamma_l)}\tau^{2j-2}} \\
& \qquad = \sum_{\rho=0}^{L}(-1)^\rho\sigma_\rho\left(\bigcup_{l=2}^{n-1}\bigcup_{\gamma\in I_l}\bigcup_{j=1}^{l-1}\zeta_j^{(\gamma)}\tau^{2j-1}\right)A_{\lambda-\rho}(x|\zeta^{(n)}|\cdots|\zeta^{(1)}|\cdots|\zeta^{(n-1\cdots21)})
\end{aligned}
\tag{5.2}
$$

where

$$
\sum_{l=1}^{n-1}\sum_{\substack{(\gamma_1,\ldots,\gamma_l)\in I_l \\ \gamma_j=k}} m(\gamma_1,\ldots,\gamma_l) = m \qquad (k=1,2,\ldots,n)
$$

and

$$
L = \sum_{l=2}^{n-1}(l-1)\sum_{\gamma\in I_l} m(\gamma).
$$

For $n = 3$, this reads as

$$
\begin{aligned}
A_\lambda^{(m)}(x|z^{(3)}|z^{(2)}|z^{(1)})|_{\bigcup_{l=1}^{2}\bigcup_{j=1}^{l} z^{(\gamma_j)}=\zeta^{(\gamma_1\cdots\gamma_l)}\tau^{2j-2}} = {} & \sum_{\rho=0}^{L}(-1)^\rho\sigma_\rho(\zeta^{(32)}\tau,\zeta^{(31)}\tau,\zeta^{(21)}\tau) \\
& \times A_{\lambda-\rho}(x|\zeta^{(3)}\tau,\zeta^{(2)}\tau,\zeta^{(1)}\tau,\zeta^{(32)}\tau,\zeta^{(31)}\tau,\zeta^{(21)}\tau).
\end{aligned}
$$

Define the polynomial for positive integer α

$$
\begin{aligned}
h_\alpha^{(m|L)}&(x|\zeta^{(n)}|\cdots|\zeta^{(1)}|\zeta^{(nn-1)}|\cdots|\zeta^{(\delta_1\cdots\delta_l)}|\cdots|\zeta^{(n-1\cdots 21)}) \\
&= x^{\alpha-2}\left\{\tau^{n(\alpha-1)}\prod_{l=1}^{n-1}\prod_{\gamma\in I_l}\prod_{j=1}^{m(\gamma)}(x\tau^{n-l}-\zeta_j^{(\gamma)}\tau^{l-1})\right. \\
&\qquad\left. -\tau^{-n(\alpha-1)}\prod_{l=1}^{n-1}\prod_{\gamma\in I_l}\prod_{j=1}^{m(\gamma)}(x\tau^{-n+l}-\zeta_j^{(\gamma)}\tau^{l-1})\right\}.
\end{aligned}
\tag{5.3}
$$

One of the most important observations is the following proposition.

Proposition 5.1. For $m(\gamma_1,\gamma_2,\ldots,\gamma_l)$ such that $\sum_{l=1}^{n-1}\sum_{(\gamma_1,\ldots,\gamma_l)\in I_l,\gamma_j=k} m(\gamma_1,\ldots,\gamma_l) = m, (k=1,2,\ldots,n)$, the polynomial $\Delta^{(m)}$ satisfies the following relation:

$$
\begin{aligned}
\Delta^{(m)}&(x_1,\ldots,x_M|z^{(n)}|z^{(n-1)}|\cdots|z^{(1)})|_{\bigcup_{l=1}^{n-1}\bigcup_{j=1}^{l} z^{(\gamma_j)}=\zeta^{(\gamma_1\cdots\gamma_l)}\tau^{2j-2}} \\
&= \sum_{1\leqslant\mu_1<\cdots<\mu_L\leqslant M}(-1)^{\sum_{i=1}^{L} M-L+i+\mu_i} \\
&\quad\times\det\left(h_{\alpha-1}^{(m)}(x_{\mu_i}|\zeta^{(n)}|\cdots|\zeta^{(1)}|\cdots|\zeta^{(n-1\cdots 21)}\right)_{1\leqslant i,\alpha\leqslant L} \\
&\quad\times\Delta(\overset{x_\mu}{\hat{x}}\ |\zeta^{(n)}|\cdots|\zeta^{(1)}|\cdots|\zeta^{(n-1\cdots 21)}).
\end{aligned}
\tag{5.4}
$$

Here

$$
\begin{aligned}
\Delta&(y_1,\ldots,y_{M-L}|\zeta^{(n)}|\cdots|\zeta^{(1)}|\cdots|\zeta^{(n-1\cdots 21)}) \\
&= \det\left(A_\lambda(y_\mu|\zeta^{(n)}|\cdots|\zeta^{(1)}|\zeta^{(nn-1)}|\cdots|\zeta^{(21)}|\cdots|\zeta^{(n-1\cdots 1)}\right)_{1\leqslant\lambda,\mu\leqslant M-L}
\end{aligned}
\tag{5.5}
$$

and

$$
L=\sum_{l=2}^{n-1}(l-1)\sum_{\gamma\in I_l}m(\gamma) \qquad M=(n-1)m-1.
$$

Proof. This follows from induction with respect to L. When $L=0$, (5.4) is obvious. Assume (5.4) for L. After another restriction, say $\zeta_{m(1)}^{(1)}=\zeta_{m(2)}^{(2)}\tau^2$, using (3.2) we have

$$
\begin{aligned}
A_{M-L}&(x|\zeta^{(n)}|\cdots|\zeta^{(1)}|\zeta^{(nn-1)}|\cdots|\zeta^{(21)}|\cdots|\zeta^{(n-1\cdots 1)})|_{\zeta_{m(1)}^{(1)}=\zeta_{m(2)}^{(2)}\tau^2} \\
&= h_1^{(m|L+1)}(x|\zeta^{(n)}|\cdots|\zeta^{(1)}|\zeta^{(nn-1)}|\cdots|\zeta^{(\delta_1\cdots\delta_l)}|\cdots|\zeta^{(n-1\cdots 21)}).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
h_\alpha^{(m|L)}(x|\zeta^{(n)}|\cdots|\zeta^{(1)}|\cdots|\zeta^{(n-1\cdots 1)})|_{\zeta_{m(1)}^{(1)}=\zeta_{m(2)}^{(2)}\tau^2} &= h_{\alpha+1}^{(m|L+1)}(x|\zeta^{(n)}|\cdots|\zeta^{(1)}|\cdots|\zeta^{(n-1\cdots 1)}) \\
&\quad -\zeta_{m(2)}^{(2)}\tau h_\alpha^{(m|L+1)}(x|\zeta^{(n)}|\cdots|\zeta^{(1)}|\cdots|\zeta^{(n-1\cdots 1)}).
\end{aligned}
$$

For other types of restrictions, we obtain similar results. Thus, using (5.2) and performing an elementary transformation, we have (5.4) for $L+1$. □

Thanks to the recursion relation for $f_\lambda^{(N)}$

$$
\begin{aligned}
f_\lambda^{(N+n-1)}&(y|z_1,\ldots,z_N,a\tau,a\tau^3,\ldots,a\tau^{2n-3}) \\
&= \sum_{\rho=0}^{n-1}(-1)^\rho\sigma_\rho(a\tau,a\tau^3,\ldots,a\tau^{2n-3})f_{\lambda-\rho}^{(N)}(y|z_1,\ldots,z_N).
\end{aligned}
$$

The polynomial $A_\lambda^{(m)}$ satisfies the following recursion relation:

$$\begin{aligned} A_\lambda^{(m)}&(x|z^{(n)}|\cdots|z^{(1)})|_{z_m^{(k)}=a\tau^{2n-2k}\quad(k=1,\ldots,n)} \\ &= (x-a\tau^{n-1})\times\sum_{\rho=0}^{n-1}(-1)^\rho\sigma_\rho(a\tau,a\tau^3,\ldots,a\tau^{2n-3})A_{\lambda-\rho}^{(m-1)}(x|z'^{(n)}|\cdots|z'^{(1)}). \end{aligned} \tag{5.6}$$

Note that for $1\leqslant\alpha\leqslant n-1$

$$A_{M_{m-1}+\alpha}^{(m-1)}(z'^{(n)}|\cdots|z'^{(1)}) = h_\alpha^{(m-1|0)}(x|z'). \tag{5.7}$$

By combining (5.6) and (5.7), and using the same argument as that in the proof for proposition 5.1, one can prove the following proposition.

Proposition 5.2. The determinant $\Delta^{(m)}$ obeys the following recursion relation:

$$\begin{aligned} \Delta^{(m)}&(x_1,\ldots,x_M|z^{(n)}|\cdots|z^{(1)})|_{z_m^{(k)}=a\tau^{2n-2k}(k=1,2,\ldots,n)} \\ &= \prod_{\mu=1}^{M}(x_\mu-a\tau^{n-1})\sum_{1\leqslant\mu_1<\cdots<\mu_{n-1}\leqslant M}(-1)^{\sum_{i=1}^{n} mn-m-n+i+\mu_i} \\ &\quad\times\det(h_\alpha^{(m-1|0)}(x_{\mu_i}|z'))_{1\leqslant i,\alpha\leqslant n-1}\Delta^{(m-1)}(x\setminus\{x_\mu\}|z') \end{aligned} \tag{5.8}$$

where we use the abbreviation $x_\mu=(x_{\mu_1},\ldots,x_{\mu_{n-1}})$.

The following are two key theorems.

Theorem 5.3. Let $P(m)$ be proposition 3.3 for $m\geqslant 1$. Then for $m>1$, $P(m)$ under the restriction $z_m^{(n)}=a,\ldots,z_m^{(2)}=a\tau^{2n-4}$, $z_{m-1}^{(1)}=a\tau^{2n-2}$ holds if $P(m-1)$ holds.

Proof. Using the relation

$$\begin{aligned} \varphi^{(m)}&(j_1,\ldots,j_{n-1})|_{z_m^{(k)}=a\tau^{2n-2k}(k=2\cdots n),z_{m-1}^{(1)}=a\tau^{2n-2}} \\ &= (-\tau)^{n-1}\frac{a-z_0\tau^2}{a\tau^{2n-2}-z_0\tau^2}\varphi^{(m-1)}(j_1,\ldots,j_{n-1}) \end{aligned}$$

the LHS of (3.6) can be expressed in a recursive way. Let us turn to the RHS. Note that

$$\begin{aligned} \tilde{h}_1^{(m|l)}&(x|z'^{(n)}|\cdots|z'^{(n-l+1)}|z^{(n-l)}|\cdots|z^{(1)}|\cdots|\zeta^{(n\cdots n-l+1)}=a) \\ &= \frac{z_0\tau^{2n-2}-a\tau^{2l-4}}{z_0-a\tau^{2n-2}}\frac{1}{x}\left\{(x\tau^{n-l}-a\tau^{l-1})\prod_{\substack{j=0\\ j\neq km(k=1,\ldots,l)}}(x\tau^{n-1}-z_j)\right. \\ &\quad\left.-(x\tau^{-n+l}-a\tau^{l-1})\prod_{\substack{j=0\\ j\neq km(k=1,\ldots,l)}}(x\tau^{-n+1}-z_j)\right\}. \end{aligned} \tag{5.9}$$

From proposition 5.2 and (5.9), we obtain

$$\begin{aligned} \tilde{\Delta}^{(m)}&(x|z^{(n)}|\cdots|z'^{(1)},z_0)|_{z_m^{(k)}=a\tau^{2n-2k}(k=2\cdots n),z_{m-1}^{(1)}=a\tau^{2n-2}} \\ &= \prod_{l=2}^{n}\frac{z_0\tau^{2n-2}-a\tau^{2l-4}}{z_0-a\tau^{2l-4}}\prod_{\mu=1}^{M}(x_\mu-a\tau^{n-1})\sum_{1\leqslant\mu_1<\cdots<\mu_{n-1}\leqslant M}(-1)^{\sum_{i=1}^{n} mn-m-n+i+\mu_i} \\ &\quad\times\det(h_\alpha^{(m-1)}(x_{\mu_i}|z'',z_0))_{1\leqslant i,\alpha\leqslant n-1}\tilde{\Delta}^{(m-1)}(x\setminus\{x_\mu\}|z'',z_0) \end{aligned} \tag{5.10}$$

where we use the abbreviation $z'' = (z'^{(n)}|\cdots|z'^{(2)}|z_1^{(1)},\ldots,z_{m-2}^{(1)})$. Thus, the claim is verified. □

Theorem 5.4. Let $P(m)$ be proposition 3.3 for $m \geqslant 1$. Then $P(m)$ under the restriction $z_m^{(n)} = z_0\tau^2,\ldots,z_m^{(2)} = z_0\tau^{2n-2}$ holds.

Proof. Under the restriction in consideration, there is only one non-zero term in the LHS of (3.6) and the only non-zero coefficient $\varphi^{(m)}$ is

$$\varphi^{(m)}(m,2m,\ldots,(n-1)m) = (-\tau)^{m(1-n)}\prod_{j=1}^{m-1}\frac{z_j'^{(1)} - z_0\tau^{2n}}{z_j'^{(1)} - z_0\tau^2}. \tag{5.11}$$

Note that in the RHS there exists a zero and a pole under the restriction. Thus, we have to first set $z_m^{(n)} = a,\ldots,z_m^{(2)} = a\tau^{2n-4}$ and reduce, then we have to set $a = z_0\tau^2$ to evaluate the RHS

$$\begin{aligned}\tilde{\Delta}^{(m)}(x|z^{(n)}|\cdots|z^{(2)}|z'^{(1)},z_0)|_{z_m^{(k)}=z_0\tau^{2n+2-2k}(k=2,\ldots,n)}\\ = (-\tau^{(n-1)})^{(n-2)}\prod_{k=2}^{n}\prod_{j=1}^{m-1}\frac{z_j^{(k)} - z_0\tau^{2n-2}}{z_j^{(k)} - z_0\tau^2}\prod_{k=2}^{n}\frac{z_j^{(1)} - z_0\tau^{2n}}{z_j^{(1)} - z_0\tau^2}\\ \times\prod_{\mu=1}^{M}(x_\mu - z_0\tau^{n-1})\sum_{1\leqslant\mu_1<\cdots<\mu_{n-1}\leqslant M}(-1)^{\sum_{i=1}^{n} mn-m-n+i+\mu_i}\\ \times\det(h_\alpha^{(m-1)}(x_{\mu_i}|z'))_{1\leqslant i,\alpha\leqslant n-1}\Delta^{(m-1)}(x\setminus\{x_\mu\}|z').\end{aligned} \tag{5.12}$$

Therefore, this proposition follows from (5.11) and (5.12). □

We now wish to show proposition 3.3. First, note that the LHS of proposition 3.3 has no singularity at $z'^{(1)} = z_0\tau^{2n-2}$. Thus, let us prove the following proposition.

Proposition 5.5.

$$\tilde{\Delta}^{(m)}(x_1,\ldots,x_M|z^{(n)}|z^{(n-1)}|\cdots|z'^{(1)},z_0)|_{z_j^{(k)}=z_0\tau^{2n-2}} = 0 \qquad (k=2,\ldots,n). \tag{5.13}$$

Proof. By the same argument as that used to prove proposition 5.2, we have an equation which can be obtained from (5.5) by replacing $A \to \tilde{A}$ and $\Delta \to \tilde{\Delta}$. Since the last row obtained in this way vanishes, the claim of this proposition is verified. □

Using a general m analogue of (4.9), we can prove the following proposition.

Proposition 5.6.

$$\begin{aligned}\lim_{z_m^{(n)}\to z_0\tau^2}\prod_{k=2}^{n}\prod_{j=1}^{m}\frac{z_j^{(k)} - z_0\tau^2}{z_j^{(k)} - z_0\tau^{2n-2}}\tilde{\Delta}(x|z^{(n)}|\cdots|z^{(2)}|z'^{(1)},z_0)\\ = \sum_{\nu=1}^{M}\prod_{\substack{\mu=1\\ \mu\neq\nu}}^{M}\frac{x_\mu - z_0\tau^{n-1}}{x_\mu - z_0\tau^{n+1}}g_1(x_\nu|z'^{(n)}|z^{(n-1)}|\cdots|z'^{(1)}|z_0)\\ \times\Delta(x_1,\ldots,\overset{\nu}{\widehat{z_0\tau^{n+1}}},\cdots,x_M|z'^{(n)}|z^{(n-1)}|\cdots|z'^{(1)}|z_0)\end{aligned}$$

where

$$g_1(x|z'^{(n)}|z^{(n-1)}|\cdots|z'^{(1)}|z_0) = \frac{z_0\tau^{n+1}}{x}\left\{\tau^{-2}\prod_{\substack{j=1\\ j\neq m}}^{N}\frac{x\tau^{n-1}-z_j}{z_0\tau^2-z_j} - \frac{x-z_0\tau^{n-1}}{x-z_0\tau^{n+1}}\prod_{\substack{j=1\\ j\neq m}}^{N}\frac{x\tau^{-n+1}-z_j}{z_0\tau^2-z_j}\right\}.$$

Now we are in a position to prove proposition 3.3.

Proof of proposition 3.3. From proposition 5.5, both sides of (3.6) are homogeneous rational functions of degree $Mm + M(M-1)/2$ with simple poles located at $z'^{(1)} = z_0\tau^2$. Taking into account the antisymmetry of the x_μ's, equation (3.6) holds if both sides of (3.6) coincide at $(n-1)m^2$ points $z^{(n)} = z_0\tau^2$ and $z^{(a)} = z^{(b)}\tau^2 (a < b)$. From proposition 5.1, both sides of (3.6), after L times' restriction, can be expressed as follows

$$\mathrm{LHS}|_{\mathrm{restrictions}} = \sum_{1\leqslant\mu_1<\cdots<\mu_L\leqslant M} \det(h^{(m)}_{\alpha-1}(x_{\mu_i}|\zeta))_{1\leqslant\alpha,i\leqslant L} S_L(x\setminus\{x_\mu\}) \tag{5.14}$$

$$\mathrm{RHS}|_{\mathrm{restrictions}} = \sum_{1\leqslant\mu_1<\cdots<\mu_L\leqslant M} \det(h^{(m)}_{\alpha-1}(x_{\mu_i}|\zeta))_{1\leqslant\alpha,i\leqslant L} S_R(x\setminus\{x_\mu\}). \tag{5.15}$$

Here, $S_L(y_1,\ldots,y_{M-L})$ and $S_R(y_1,\ldots,y_{M-L})$ are skew symmetric with respect to $(y_1,\ldots,y_{M-L})$. Note that the degrees of $S_L(y_1,\ldots)$ and $S_R(y_1,\ldots)$ with respect to y_1 are $M-L+m-1$. Furthermore, $\det(h^{(m)}_{\alpha-1}(y_1,y_2,\ldots,y_L|\zeta))_{1\leqslant\alpha,j\leqslant L} = cy_1^{M+m-1}y_2^{M+m-2}\cdots y_L^{M+m-L}+\cdots$. By comparing the coefficients of the $\mathrm{LHS}|_{\mathrm{restrictions}}$ and $\mathrm{RHS}|_{\mathrm{restrictions}}$ with respect to $x_1^{M+m-1}x_2^{M+m-2}\cdots x_L^{M+m-L}$, we conclude that the $\mathrm{LHS}|_{\mathrm{restriction}} = \mathrm{RHS}|_{\mathrm{restriction}}$ if and only if

$$S_L(x_1,\ldots,x_{M-L}) = S_R(x_1,\ldots,x_{M-L}). \tag{5.16}$$

Consequently, taking into account the antisymmetry of the x_μ's, we have only to examine $\mathrm{LHS}|_{\mathrm{restriction}} = \mathrm{RHS}|_{\mathrm{restriction}}$ at $(n-1)m^2 - Lm$ planes. After repeating this procedure, equation (3.6) reduces to theorem 5.3 and 5.4. For simplicity and clarity, we first demonstrate how this induction scheme works when $n = 3$.

For $3 \geqslant a > b \geqslant 1$, let $m(a,b)$ stand for the number of restrictions $z^{(b)} = z^{(a)}\tau^2$ and let $0 \leqslant l \leqslant 1$ be the number of restrictions $z^{(3)} = z_0\tau^2$. We denote such restrictions by $(l, m(2,1)+m(3,2))$ and introduce a lexicographical order by $(1,m) > (1,m-1) > \cdots > (1,0) > (0,m) > \cdots > (0,0)$. The restriction (l,k) is larger than (l',k') if $(l,k) > (l',k')$. We wish to show that $(l, m(2,1)+m(3,2))$ reduces to larger restrictions and $(1,m)$ reduces to theorem 5.3 and 5.4.

First step. Set $(l, m(2,1)+m(3,2)) = (1,m)$. In this case we need $m^2 - m(1+m(3,1))$ planes on which (5.16) holds for $L = m+1+m(3,1)$. We have m restrictions which reduce to theorem 5.4 and $m(m-1) - m(2,1)m(3,2)$ restrictions which reduce to theorem 5.3 and $(m-1-m(3,1)-m(3,2))(m-1-m(3,1)-m(2,1))$ restrictions which reduce to the previous two restrictions. Hence, we actually have $m^2 - m(1+m(3,1)) + (m(3,1)+1)^2$ planes.

Second step. Set $(l, m(2,1)+m(3,2)) = (1,k)$ where $k < m$. In this case, we need $2m^2 - m(1+k+m(3,1))$ planes on which (5.16) holds. We have m restrictions which reduce to theorem 5.4, $k(m-1) - m(3,2)m(2,1)$ restrictions which reduce to theorem 5.3 and $(m-k)(2m-k-2)$ restrictions which reduce to larger restrictions. We also have $(m-1-m(3,1)-m(3,2))(m-1-m(3,1)-m(2,1))$ additional restrictions at $z^{(1)} = z^{(3)}\tau^2$ which reduces to the previous three restrictions. Hence, we have $2m^2 - m(1+k+m(3,1)) + (k-m+1+m(3,1)/2)^2 + m(3,1)(3m(3,1)+4)/4$ planes.

Third step. We can show in a way similar to that used in the second step that this induction scheme works for $l = 0$.

Therefore, *equation* (3.6) *is proved for* $n = 3$. Next, we show (3.6) for general n.

Let l denote the maximal length of restrictions such that $z^{(n+1-l)} = z^{(n+2-l)}\tau^2 = \cdots = z^{(n)}\tau^{2l-2} = z_0\tau^{2l}$. Let $r_j (1 \leqslant j \leqslant n-2)$ denote the number of restrictions of type $z^{(k)} = z^{(k+1)}\tau^2 = \cdots = z^{(k+j)}\tau^{2j}$ where $1 \leqslant k \leqslant n-j$. Set $L = \sum_{j=1}^{n-2} jr_j$. We introduce the lexicographical order in $\{(l, L, r_{n-2}, \ldots, r_1) \mid 0 \leqslant L + l \leqslant (n-2)m + 1\}$ by

$$\begin{aligned}(n-2, (n-2)(m-1)+1, m-1, 0, \ldots, 0, 1) \\ &> (n-2, (n-2)(m-1)+1, m-2, 1, 0, \ldots, 0, 2) \\ &> \cdots > (0, 1, 0, \ldots, 0, 1) > (0, 0, 0, 0, \ldots, 0).\end{aligned}$$

At the stage of induction of type $(l, L, r_{n-2}, \ldots, r_1)$, *we need more than or equal to* $(n-1)m^2 - m(l+L)$ restriction planes.

Remark 1. In the expression for (5.1), let us call the z, which belongs to one of the $\zeta^{(i)}$, *free* z. Then we have at least $m-2$ free z's.

Remark 2. There are restrictions which are not counted by r_j $(j = 1, \ldots, n-2)$, e.g. restrictions counted by $m(a, b)$ where $a - b > 1$. Let us call such a restriction a *bad restriction.* Even if we need less than or equal to m bad restrictions to obtain $(n-1)m^2 - m(l+L)$ restriction planes for $(l, L, r_{n-2}, \ldots, r_1)$, our induction scheme does work because, after a bad restriction, we still have the same number of not bad restrictions as that for $(l, L, r_{n-2}, \ldots, r_1)$, while we need only $(n-1)m^2 - m(l+L) - m$ restriction planes.

First step. Set $(l, L, r_{n-2}, \ldots, r_1) = (n-2, (n-2)(m-1)+1, m-1, 0, \ldots, 0, 1)$. Then, we need $m^2 - m$ planes on which (5.16) holds. Let us first consider $m(n, \ldots, 2) = m-1$, $m(2, 1) = 1$, $m(1) = m-2$ and the other $m(\gamma) = 0$. We have m planes which reduce *to theorem* 5.4 and $(m-1)^2$ *planes which reduce to theorem* 5.3. Thus, we actually have $m^2 - m + 1$ planes. Next, consider $m(n, \ldots, 2) = m-2$ and $m(n-1, \ldots, 1) = 1$. In this case, we have m planes which reduce to theorem 5.4 and $(m-1)^2 - (m-2)$ planes which reduce to theorem 5.3. We also have $m-2$ bad restrictions. Hence, from remark 2, this case reduces to theorems 5.3 and 5.4. In general, when we replace $(m(n, \ldots, 2), m(n-1, \ldots, 1)) = (m-p, p-1)$ by $(m(n, \ldots, 2), m(n-1, \ldots, 1)) = (m-1-p, p)$, we need $m - 2p$ additional bad restrictions. Thus, every $(l, L, r_{n-2}, \ldots, r_1)$ reduces to theorems 5.3 and 5.4. In a similar way, one can show that for other restrictions of type $(l, L, r'_{n-2}, \ldots, r'_1)$, we have $m^2 - m + 1$ planes, while we need $m^2 - m$ planes.

Second step. Set $(l, L) = (n-2, (n-2)(m-1))$. Such a restriction can be obtained by dividing one of the chains

$$z^{(k)} = z^{(k+1)}\tau^2 = \cdots = z^{(k+j)}\tau^{2j} \tag{5.17}$$

of restriction type $(l, L+1)$ into two pieces like

$$z^{(k)} = \cdots = z^{(k+i)}\tau^{2i} \qquad \text{and} \qquad z^{(k+i+1)}\tau^{2(i+1)} = \cdots = z^{(k+j)}\tau^{2j} \tag{5.18}$$

where $0 \leqslant i \leqslant j-1$.

Note that there are two kinds of restriction planes for (l, L), *old ones* and *new ones*: old ones are restriction planes which decrease the degree of (3.6) even for (5.17); and new ones are those which do not decrease the degree of (3.6) for (5.17) but do decrease it for (5.18).

For example, the following restriction of type (l, L):

$$\begin{aligned} z_1^{(3)} &= z_1^{(4)}\tau^2 = \cdots = z_1^{(n)}\tau^{2(n-3)} = z_0\tau^{2(n-2)} \\ z_i^{(2)} &= z_i^{(3)}\tau^2 = \cdots = z_i^{(n)}\tau^{2(n-2)} \qquad 2 \leqslant i \leqslant m-1 \\ z_m^{(2)} &= z_m^{(3)}\tau^2 = \cdots = z_m^{(n-1)}\tau^{2(n-3)} \\ z_1^{(1)} &= z_1^{(2)}\tau^2 \end{aligned} \tag{5.19}$$

can be obtained from the following restriction of type $(l, L+1)$:

$$\begin{aligned} z_1^{(3)} &= z_1^{(4)}\tau^2 = \cdots = z_1^{(n)}\tau^{2(n-3)} = z_0\tau^{2(n-2)} \\ z_i^{(2)} &= z_i^{(3)}\tau^2 = \cdots = z_i^{(n)}\tau^{2(n-2)} \qquad 2 \leqslant i \leqslant m \\ z_1^{(1)} &= z_1^{(2)}\tau^2 \end{aligned} \tag{5.20}$$

by resetting the relation between $z_m^{(n-1)}$ and $z_m^{(n)}$. For (5.20), we have $m^2 - m + 1$ restriction planes

$$\begin{aligned} z_j^{(2)} &= z_1^{(3)}\tau^2 \qquad 1 \leqslant j \leqslant m \\ z_k^{(1)} &= z_1^{(2)}\tau^2 \qquad 2 \leqslant i, \leqslant m, 1 \leqslant k \leqslant m-1 \end{aligned} \tag{5.21}$$

which reduce to theorems 5.3 and 5.4. For (5.19), $m^2 - m + 1$ restriction planes (5.21) are old. On the other hand, restriction planes $z_m^{(n-1)} = z_m^{(n)}\tau^2$, $z_1^{(2)} = z_m^{(n)}\tau^2$ and $z_k^{(1)} = z_m^{(n)}\tau^2 (1 \leqslant k \leqslant m-1)$ are new.

It is evident that the number of restriction planes for $(l, L+1)$ and the number of old restriction planes for (l, L) coincide. Thus, in order to show this case, we need $m^2 - (m^2 - m + 1) = m - 1$ new restriction planes.

Let us consider the case $i = 0$ in (5.18). Now there are p free z's belonging to $\zeta^{(k)}$ and $m - 1 - p$ free z's belonging to $\zeta^{(i)}$ where $i \neq k$ and $1 \leqslant p \leqslant m-1$. Thus, we have at least $p + p(m-1-p) \geqslant m-1$ new restriction planes. One can show the other case $1 \leqslant i \leqslant j-1$ similarly. Hence, this reduces to larger restrictions.

Third step. Suppose we have a sufficient number of restriction planes to prove (5.16) for a restriction of type $(n-2, L, r_{n-2}, \ldots, r_1)$ where $L \leqslant (n-2)(m-1)$. If we replace L by $L-1$, we need m new restriction planes. On the other hand, there are now at least m free z's. Thus, by an argument parallel to that given in the second step, this case reduces to larger restrictions.

Fourth step. Suppose that we have a sufficient number of restriction planes to prove (5.16) for the restriction of type $(l, L, r_{n-2}, \ldots, r_1)$ where $L \leqslant (n-2)(m-1)$. If we replace l by $l-1$, we need m new restriction planes. Now we have at least m free z's. Thus, from the same argument given in the second step, this case reduces to larger restrictions. As for the restriction obtained by replacing L by $L-1$, the argument again perfectly parallels the one given in the second step.

Therefore, after repeating this procedure, we finally reach

$$(l, L, r_{n-2}, \ldots, r_1) = (0, \ldots, 0)$$

and this is what we wish to prove. □

Next we note that proposition 3.3 implies (1.7) for $\varepsilon_1 = i$. For that purpose, let us seek the residue formula for $\overline{G}$ defined below.

Let

$$G(z^{(n)}|\cdots|z^{(1)}) = \oint_C \mathrm{d}x_1 \cdots \oint_C \mathrm{d}x_M\, \overline{G}(x|z^{(n)}|\cdots|z^{(1)})\Psi(x|z)$$

and

$$\overline{G}_i(z^{(n)}|\cdots|z^{(1)}) = G^{\overbrace{n\cdots n}^{m}\cdots\hat{i}\cdots\hat{i}\cdots\overbrace{1\cdots 1}^{m}\overbrace{i\cdots i}^{m}}(z^{(n)}|\cdots|z^{(1)}).$$

Then, we have

$$\overline{G}_i^{(m)}(x|z) = (-\tau)^{i-1} \sum_{1\leqslant\mu_1<\cdots<\mu_{n-1}\leqslant M} (-1)^{\sum_{i=1}^{n} mn-m-n+i+\mu_i}$$
$$\times \det(h_\alpha^{(m-1)}(x_{\mu_i}|z'))_{1\leqslant i,\alpha\leqslant n-1} \overline{G}_i^{(m-1)}(\overset{\mu}{\hat{x}}\,|z). \qquad (5.22)$$

Therefore, (3.4) for $i \neq 1$ reduces to proposition 3.3.

Thus, we have proved theorem 3.1.

6. Conclusions and discussions

In this paper, we have presented an integral formula for the level-zero quantum KZ equation associated with the vector representation of $U_q(\widehat{\mathfrak{sl}_n})$. This work is a generalization of our previous paper [1]. Smirnov obtained the formula for form factors of the $SU(n)$ chiral Gross–Neveu model in appendix A of his book [3]. It gives a rational scaling limit of the present work (i.e. $q = \mathrm{e}^{\epsilon}$, $z = \mathrm{e}^{-\epsilon n(\beta/\pi \mathrm{i})}$, $\epsilon \to 0$). Smirnov [3] has studied form factors of integrable massive theories; equations (1.1) and (1.2) are two of the axioms proposed by him that form factors obey. Let $G(z)$ be a form factor of a certain operator. Then a ϑ function in the integral kernel should be determined. It is interesting and important to solve this problem.

Finally, we discuss related works on the integral formulae for quantum KZ equations. In [5], an integral formula for correlation functions of the XXZ model was obtained on the basis of the bosonization of the level-one highest-weight representations of $U_q(\widehat{\mathfrak{sl}_2})$. This scheme gives only one particular solution though these correlations satisfy the quantum KZ equation of arbitrary level. A formula for a higher spin analogue of the XXZ model in terms of a Jackson-type integral was given in [6] by using the level-k bosonization of $U_q(\widehat{\mathfrak{sl}_2})$.

In [7, 8], solutions using Jackson-type integrals are obtained. These formulae are, in principle, valid for general levels, as opposed to our integral formula which is restricted to level zero. On the other hand, the problem of choosing the cycles for Jackson-type integrals, which accommodate the freedom of the solutions, has not been studied much.

Tarasov and Varchenko [9] improved the Jackson-type integral formula for $U_q(\mathfrak{gl}_n)$, such that the solutions automatically satisfy the R-matrix symmetry. The number of Jackson-type integrals is $n(n-1)m/2$ for $N = nm$, whereas our formula is written by a $(n-1)m-1$ fold integral. For $n = 2$, two numbers are different by only 1; however, the difference increases as n increases.

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